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Global Solutions of the Boltzmann Equation on a Lattice

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The nonlinear Boltzmann equation with a discretized spatial variable is studied in a Banach space of absolutely integrable functions of the velocity variables. Conservation laws and positivity are utilized to extend weak local solutions to a global solution. This is shown to be a strong solution by analytic semigroup techniques.

KEY WORDS: Boltzmann equation; partial differential equations; global solutions; mild solutions; lattice; semigroup.

1. INTRODUCTION

Although the Boltzmann equation was first derived over 100 years ago,[4] the existence of solutions to the full nonlinear equation is barely understood. The first significant paper seems to have been written by Carleman,(2) who treated hard-sphere molecules in the spatially homogeneous case. Grad(3) presented an existence proof for the spatially dependent case for small times, and a global solution for the spatially homogeneous case for "modified Maxwell molecules," i.e., molecules interacting through an inverse-fifth-power law with a cutoff at some finite range. (The inverse fifth power leads to a collision cross section which is independent of the relative speed of the colliding molecules, and the cutoff gives a finite collision rate.) In both Refs. 2 and 3, incidentally, the initial distribution was, to some extent, restricted.

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[4] For an historical review see Ref. 1.

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This lack of generality is crucial to Carleman's estimates and Grad's treatment of the spatially dependent case.

Morgenstern[4] obtained a global solution for the spatially dependent case by introducing a "mollifying kernel" into the collision term, as is done, for example, in treating infrared divergences in quantum field theory.[5] Again, the popular "modified Maxwell" case was considered. This mollifier removes the difficulty one encounters in trying to norm products like $\varphi(x, v)\varphi(x, v')$ in an L_1 space.

More recent studies have dealt with systems close to equilibrium,[6–9] while Povzner[10] has considered a mollifier similar to that of Ref. 4. Much of the recent work (e.g., Ref. 7) employs norm estimates in spaces whose physical relevance is difficult to envision, and in our work as reported here we attempt to avoid this by working in a simple L_1 space. Also, we place minimal restrictions on the initial data, unlike some of the earlier work mentioned above. In particular, our initial data can be arbitrarily far from equilibrium. On the debit side of our ledger is the fact that we are unable to deal with other than modified Maxwell molecules; in particular, our present techniques cannot be used for hard spheres, although we hope to deal with that case in a subsequent publication, perhaps by introducing a velocity cutoff and proving the existence of a limit, or by suitably restricting the initial data, as was done, for example, in Ref. 2.

More importantly, our Boltzmann equation is set on a (periodic) lattice, i.e., we replace the gradient term by a finite difference approximation. In this way we avoid the introduction of the mollifier.

On the credit side, our proof, as mentioned above, is essentially independent of the initial distribution, and applies for all times. Again, we plan to consider in a subsequent paper the question of the limit as the lattice spacing tends to zero, following methods which are familiar in field theory.[5]

The reader intent on studying the details of the collision term in the Boltzmann equation and its properties should consult Ref. 3, or a kinetic theory text (e.g., Ref. 11). Our basic methods are adaptations of techniques described by Reed and Simon[12] for dealing with nonlinear differential equations. The reader should consult that reference and Ref. 14 for relevant mathematical background.

2. FORMULATION OF THE INITIAL VALUE PROBLEM

We consider the equation

$$\frac{\partial \Psi_i}{\partial t}(c, t) + (A, \Psi)_i(c, t) = G(\Psi, \Psi)_i - D(\Psi)_i \Psi_i \tag{1}$$

Here the index i is the spatial index denoting the ith lattice point in the periodic three-dimensional cube Λ^3, while c is the (dimensionless) velocity vector. The operators A, G, and D are defined below. We seek solutions in the Banach space

$$B = \bigoplus_{i=1}^{n^3} L^1(\mathbb{R}^3)$$

where n^3 represents the number of lattice sites. In B we have

$$\|\Psi\| = \sum_{i=1}^{n^3} \int_{-\infty}^{\infty} |\Psi_i(c)| \, dc \tag{2}$$

(When the meaning is obvious, we shall suppress the spatial variable i.) We denote by $\mathscr{T}_+$ the cone of positive functions in B, and by $\mathscr{G}(\mathscr{T}_+)$ the cone of measurable functions $\Psi(\cdot)\colon \mathbb{R}_+ \to \mathscr{T}_+$.

The operator A with domain $\mathscr{D}(A)$ is the finite difference approximation to the gradient term. To give A specifically, let π be an identification between the lattice Λ^3 and the first n^3 positive integers. Then A is an $n^3 \times n^3$ matrix:

$$A_{ij} = \sum_{\hat{u}} (c \cdot \hat{u}) \Delta_{ij}^{\hat{u}}(c) \tag{3}$$

where

$$\Delta_{ij}^{\hat{u}}(c) = \delta_{ij} - \delta_{i,\pi(\pi^{-1}(j)+\hat{u})}, \qquad c \cdot \hat{u} > 0, \qquad \Delta^{\hat{u}}(-c) = \Delta^{\hat{u}}(c)^* \tag{4}$$

and the sum is over the three orthogonal coordinate vectors $\hat{u}$. The periodic boundary conditions are imposed by viewing the lattice as a three-dimensional torus, and thus $\pi^{-1}(j) + \hat{u} \in \Lambda^3$ for every j.

A convenient representation of A can be obtained in terms of tensor products. Suppose first that $c_x, c_y, c_z \geqslant 0$. Consider the $n \times n$ matrix E defined by

$$E_{ij} = \begin{cases} \delta_{n,j}, & i = 1 \\ \delta_{i,j+1}, & i > 1 \end{cases}$$

Then the lattice sites can be numbered such that

$$\begin{aligned} A = (c_x + c_y + c_z) I \otimes I \otimes I - c_x(E \otimes I \otimes I) \\ - c_y(I \otimes E \otimes I) - c_z(I \otimes I \otimes E) \end{aligned} \tag{5}$$

(Here I is the $n \times n$ identity matrix.) We observe $E^n = I$. Further, if any $c_i < 0$, the corresponding representation of A is obtained from Eq. (4), i.e., $E \to E^*$.

The collision terms G and D are, by virtue of the assumed collision model, bounded bilinear (resp. linear), positivity-preserving functionals $G \in \mathscr{L}(B \times B, B)$ and $D \in \mathscr{L}(B, L^\infty(\mathbb{R}^3))$, with the invariance property

$$\int dc \, I(c)\{G(\Psi, \Psi) - D(\Psi)\Psi\} = 0$$

for $I(c)$ a "collision invariant"[3,11] and $\Psi \in B$. We denote in the usual way the operator norms $\|G\|$ and $\|D\|$.

Finally, a solution of Eq. (1) is defined to be a strongly differentiable map $\Psi: R_+ \to B$ satisfying a specified initial condition $\Psi(0) = \varphi_0 \in \mathscr{D}(A) \cap \mathscr{T}_+$. Our method of proof will be to convert Eq. (1) to a pair of integral equations, Eqs. (6) and (7) below.

These equations are the same as those used by most of the authors quoted previously, although our approach is novel in that we utilize both equations for a single existence proof. [For example, Grad[3] uses Eq. (6) to deal with the spatially dependent case and Eq. (7) for the homogenous case; but see Arkeryd,[13] who also uses two equations for treating the spatially homogeneous case.]

The existence of a solution to Eq. (6) for sufficiently small time will be demonstrated using standard iterative techniques. We then demonstrate that any solution of Eq. (6) is also a solution of Eq. (7). Again, iterative techniques are required. Moreover, solutions of the latter are positive, and thus the solution of Eq. (6) is positive. This fact, along with the invariance property written above, suffices to extend the local solution to all time. Unicity appears as a corollary.

The first integral equation is obtained from Eq. (1) by treating A as the generator of a semigroup:

$$\Psi(t) = e^{-tA}\varphi_0 + \int_0^t e^{-(t-s)A}\{G(\Psi(s), \Psi(s)) - D(\Psi(s))\Psi(s)\}\, ds \tag{6}$$

The second integral equation is obtained by treating $D(\Psi)\Psi$ as a perturbation of A:

$$\Psi(t) = T_\Psi(t, 0)\varphi_0 + \int_0^t T_\Psi(t, t')G(\Psi(t'), \Psi(t'))\, dt' \tag{7}$$

where $T_\Psi(t, s)\xi_0$ is a solution of the homogeneous equation

$$d\varphi/dt + A\varphi + D(\Psi)\varphi = 0$$

satisfying $\varphi(s) = \xi_0$. (In physicist's jargon, T_Ψ is a time-ordered exponential.)

3. SEMIGROUP PROPERTIES AND THE ITERATIVE SCHEME

We begin by studying the properties of the two-parameter evolution operator $T_\Psi(t_2, t_1)$ and of the semigroup

$$U(t) = e^{-tA}, \qquad t \in \mathbb{R}_+ \tag{8}$$

We collect the main results in Theorem 1.

Theorem 1. (a) $U(t)$ and $T_\Psi(t_2, t_1)$ are invariant on the cone of positive functions $\mathscr{T}_+ \subset B$ for t and $t_2 - t_1$ positive, and $\Psi \in \mathscr{G}(\mathscr{T}_+)$.

(b) $U(t)$ is a contraction semigroup and continues analytically to a bounded holomorphic semigroup $U(z)$.

(c) $T_\Psi(t_2, t_1)$ is a contraction mapping on B for $t_2 - t_1 > 0$ and $\Psi \in \mathscr{G}(\mathscr{T}_+)$.

Proof. Noting that A^m is bounded on the subspace of functions in B with support in a fixed, compact $K \subset \mathbb{R}^3$, we may represent the exponential e^{-tA} by its power series on the dense linear manifold M_0 of functions in $\mathscr{T}_+$ with compact support; in fact, M_0 is an invariant domain for A and hence for $U(t)$. We note that M_0 can be written as the infinite union of subspaces M_N of functions with support in the hypercube about the origin with sides of length $2N$. Since the off-diagonal terms of $-A$ are positive for a fixed power of A, say A^l, if $(-A^m)_{ki} = 0$, $m \leqslant l$, then $(-A^{l+1})_{ki} \geqslant 0$. Therefore, every element of e^{-tA} is a power series in $t|c\cdot\hat{u}|$ with positive coefficient to lowest order. Hence, for $t|c\cdot\hat{u}|$ sufficiently small, e^{-tA} has nonnegative entries, and $(e^{-tA}f)_i \geqslant 0$ for $f \in \mathscr{T}_+ \cap M_N$. But since $e^{-2tA} = e^{-tA}e^{-tA}$, the result extends to arbitrary t. Therefore $U(t)\mathscr{T}_+ \cap M_N \subset \mathscr{T}_+ \cap M_N$ and thus $U(t)\mathscr{T}_+ \subset \mathscr{T}_+$.

On M_N, $\mathscr{A}(t) = -A - D(\Psi(t))$ is a bounded operator, and $T_\Psi(s, t)$ is given explicitly by the limit[14]

$$T_\Psi(t, s) = s\text{-}\lim_{m\to\infty} \exp\int_{t_{m-1}}^{t_m} \mathscr{A}(t')\,dt' \exp\int_{t_{m-2}}^{t_{m-1}} \mathscr{A}(t')\,dt' \cdots \exp\int_{t_0}^{t_1} \mathscr{A}(t')\,dt' \tag{9a}$$

where the limit is taken over n-partitions $t = t_m > t_{m-1} > \cdots > t_1 > t_0 = s$. Using the Lie product formula and the uniform boundedness of the

$$\exp\int_{t_i}^{t_{i+1}} \mathscr{A}(t')\,dt',$$

we can represent $T_\Psi(t, s)$ as a double limit

$$\begin{aligned} T_\Psi(t, s) = s\text{-}\lim_{m\to\infty}\lim_{n\to\infty}\Bigg\{&\left[U\!\left(\frac{t_m - t_{m-1}}{n}\right)\exp\frac{-\int_{t_{m-1}}^{t_m} D(\Psi(s))\,ds}{n}\right]^n \cdots \\ &\times\left[U\!\left(\frac{t_1 - t_0}{n}\right)\exp\frac{-\int_{t_0}^{t_1} D(\Psi(s))\,ds}{n}\right]^n\Bigg\} \end{aligned} \tag{9b}$$

But D is diagonal and positive on $\mathscr{T}_+$, and therefore so is

$$\exp\left\{-\int_{t_i}^{t_{i+1}} D(\Psi(s))\,ds\right\}$$

Thus $T_\Psi(t, s)\mathscr{T}_+ \subset \mathscr{T}_+$, completing the proof of (a).

To prove (b), we note that the streaming operator A has the property that

$$\sum_{i=1}^{n^3} (A^m)_{ij} = 0 \tag{10}$$

for any $m \in Z_+$. By a simple calculation, $U(t)$ is seen to be isometric on $\mathscr{T}_+$. But a $\mathscr{T}_+$ invariant contractive linear map S is a contraction on B. For, by decomposing $f \in B$ as $f = f_1 - f_2, f_i \in \mathscr{T}_+$, it is evident that

$$\|Sf\| \leqslant \|Sf_1\| + \|Sf_2\| \leqslant \|f_1\| + \|f_2\| = \|f\|$$

Thus $U(t)$ is contractive.

To prove that $U(t)$ continues to a bounded holomorphic semigroup we first consider the case $c_i > 0$ and derive an explicit representation for

$$U(\xi) = U_x(\xi) \otimes U_y(\xi) \otimes U_z(\xi) \tag{11}$$

If we define $E_1 = E$ and $E_\alpha = E_{\alpha-1}E$, then $E_{n+\alpha} = E_\alpha$ and $E_n = I = E_0$. Therefore, setting $c_x\xi = s$,

$$e^{sE} = \sum_{k=0}^{\infty} \frac{1}{k!} s^k E^k = \sum_{\alpha=0}^{n-1} \left(\sum_{p=0}^{\infty} \frac{s^{np+\alpha}}{(np+\alpha)!} \right) E_\alpha \equiv \sum_{\alpha=0}^{n-1} f_\alpha(s) E_\alpha \tag{12}$$

Let w_α be a primitive nth root of unity:

$$w_\alpha = e^{2\pi i\alpha/n}$$

Then

$$e^{sw_\alpha} = \sum_{\beta=0}^{n-1} f_\beta(s) w_\alpha \tag{13}$$

Thus the coefficients f can be found from Eq. (13) and substituted into (12). One easily obtains

$$f_\alpha(s) = \frac{1}{n} \left(\sum_{l=1}^{n} w_{-\alpha l} e^{sw_l} \right)$$

and

$$e^{s(-I+E)} = \frac{1}{n} \sum_{\alpha=0}^{n-1} \sum_{l=1}^{n} e^{s(w_l - 1)} w_{-\alpha l} E_\alpha \tag{14}$$

To verify that $U_x(s)$ is a bounded holomorphic semigroup, it is sufficient to show[15] that $U_x(s)$ and $sAU_x(s)$ are bounded uniformly in a sector $\mathscr{S}_\theta \subset \mathbb{C}$,

$$\mathscr{S}_\theta = \{z \mid |\arg z| < \theta < \pi/2\}$$

Write $s = u + iv$ and $w_l = \cos\theta_l + i\sin\theta_l$. Then we see from Eq. (13) that $U_x(s)$ will be uniformly bounded for

$$\frac{u}{v} \geqslant \frac{\sin\theta_l}{\cos\theta_l - 1} = -\operatorname{ctn}\frac{\theta_l}{2}$$

an inequality which can always be satisfied for positive u. (A completely analogous computation gives the same result if $c_x < 0$). The uniform boundedness of $U(t)$ now follows from Eq. (11).

The uniform boundedness of $sAU_x(s)$, and hence of $sAU(s)$, is an immediate consequence of the boundedness of the function $g(\xi) = \xi e^{-\xi}$, $\operatorname{Re}\xi \geqslant 0$. This proves part (b).

Finally, $\exp\{-\int D(\Psi(s))\,ds\}$ is clearly contractive, since D is positive. Again by the representation, Eq. (9b), $T_\Psi(t_2, t_1)$ is the limit of a product of contractions, proving (c) and thus completing the proof of the theorem.

Note that on M_0, $T_\Psi(t_2, t_1)$ is given explicitly by the Dyson series representation of the time-ordered exponential

$$T_\Psi(t_2, t_1) = I + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, \mathscr{A}(t_1) \cdots \mathscr{A}(t_n)$$

Corollary 2. For all $f \in B$,

$$\sum_{i=1}^{n^3} (U(t)f)_i = \sum_{i=1}^{n^3} f_i$$

We remark that this corollary shows (loosely speaking) that our lattice approximation preserves translational invariance. This turns out to be a key element in our existence proof. We now solve Eq. (6) by iteration, defining

$$\Psi_0(c,t) = \varphi_0(c) \tag{15a}$$

and

$$\Psi_n(c, t) = U(t)\varphi_0(c) + \int_0^t U(t - s)J(\Psi_{n-1})(s)\,ds \tag{15b}$$

where we have defined

$$J(\Psi) = G(\Psi, \Psi) - D(\Psi)\Psi \tag{15c}$$

Proposition 3. For t sufficiently small, $\|\Psi_n(t)\| \leqslant M$, independent of t and n.

In fact, the proof is immediate. For, $\|\Psi_n\| \leqslant \|\varphi_0\| + t\|J\|\,\|\Psi_{n-1}\|^2$, and the result follows for $t < 1/4\|J\|\,\|\varphi_0\|$.

Likewise, the sequence $\{\Psi_n\}$ may be shown to be Cauchy by the estimate

$$\|\Psi_{n+1} - \Psi_n\| \leqslant t\|J(\Psi_n) - J(\Psi_{n-1})\| \leqslant 2t\|J\|M\|\Psi_n - \Psi_{n-1}\|$$

Continuous dependence of the fixed point on the inital datum φ_0 is immediate. Thus, we have the following result:

Proposition 4. The iterative scheme (15) converges to a solution $\Psi(t)$ of Eq. (6) for $t < \min\{1/(4\|J\|\,\|\varphi_0\|),\, 1/(2\|J\|M)\}$ and $\Psi(t)$ is a continuous function of the initial datum φ_0.

Our next step is to prove that a positive solution exists to Eq. (7), again for sufficiently small time. Define the iterative scheme

$$\Psi_0(c, t) = \varphi_0(c) \tag{16a}$$

$$\Psi_{n+1}(c, t) = T_{\Psi_n}(t, 0)\varphi_0(c) + \int_0^t T_{\Psi_n}(t, t')G(\Psi_n(t'))\,dt' \tag{16b}$$

where we have written $G(\Psi(t)) \equiv G(\Psi(t), \Psi(t))$. We have as before the result:

Proposition 5. For t sufficiently small, $\|\Psi_n(t)\| \leqslant M_1$ independent of t and n.

Now, to prove the sequence $\{\Psi_n\}$ is Cauchy, let us define

$$\Psi_{n+1/2}(c, t) = T_{\Psi_{n-1}}(t, 0)\varphi_0(c) + \int_0^t T_{\Psi_{n-1}}(t, t')G(\Psi_n(t'))\,dt'$$

and write $\underset{s}{\overset{t}{\mathrm{S}}}$ for $\sup_{0 \leqslant s \leqslant t}$. Then from the boundedness of G and Theorem 1(b), we have immediately the estimate

$$\begin{aligned} \|\Psi_{n+1/2}(t) - \Psi_n(t)\| &\leqslant t\|G\| \underset{s}{\overset{t}{\mathrm{S}}} \|\Psi_n(s) - \Psi_{n-1}(s)\|(\|\Psi_n\| + \|\Psi_{n-1}\|) \\ &\leqslant 2t\|G\|M_1 \underset{s}{\overset{t}{\mathrm{S}}} \|\Psi_n(s) - \Psi_{n-1}(s)\| \end{aligned}$$

for t sufficiently small. Using Theorem 1(b) again, we obtain

$$\begin{aligned} \|\Psi_{n+1}(t) - \Psi_{n+1/2}(t)\| \leqslant{}& \|T_{\Psi_n}(t, 0) - T_{\Psi_{n-1}}(t, 0)\|\,\|\varphi_0\| \\ &+ t \underset{s}{\overset{t}{\mathrm{S}}} \|T_{\Psi_n}(t, s) - T_{\Psi_{n-1}}(t, s)\|\,\|G\|\,\|\Psi_n(s)\|^2 \end{aligned}$$

In order to estimate $\|T_{\Psi_n} - T_{\Psi_{n-1}}\|$, let us define

$$\chi(t) = (T_{\Psi_n}(t, s) - T_{\Psi_{n-1}}(t, s))\xi_0$$

for fixed s. Then χ is the solution of the coupled system

$$\frac{d\Psi}{dt} + [A + D(\Psi_n(t))]\Psi = 0, \qquad \Psi(s) = \xi_0$$

$$\frac{d\chi}{dt} + [A + D(\Psi_{n-1}(t))]\chi = D(\Psi_{n-1}(t) - \Psi_n(t))\Psi(t), \qquad \chi(s) = 0$$

We may write

$$\chi(t) = \int_0^t T_{\Psi_n}(t, s)D(\Psi_{n-1}(s) - \Psi_n(s))\Psi(s)\, ds$$

But then

$$\|\chi\| \leqslant t\|D\| \overset{t}{\underset{0}{S}} \|\Psi_n(s) - \Psi_{n-1}(s)\| \, \|\xi_0\|$$

Thus,

$$\|\Psi_{n+1}(t) - \Psi_{n+1/2}(t)\| \leqslant \|D\|(t\|\varphi_0\| + t^2\|G\|M_1^2) \overset{t}{\underset{s}{S}} \|\Psi_n(s) - \Psi_{n-1}(s)\|$$

Collecting these results, it is sufficient to assume $0 \leqslant t \leqslant T_0$, for $T_0^{-1} = 8 \|G\|M_1 + 8M_1 + 5\|D\|\|\varphi_0\|$ to obtain

$$\|\Psi_{n+1/2}(t) - \Psi_n(t)\| \leqslant (1/4)\overset{t}{\underset{s}{S}} \|\Psi_n(s) - \Psi_{n-1}(s)\|$$

$$\|\Psi_{n+1}(t) - \Psi_{n+1/2}(t)\| \leqslant (1/4)\overset{t}{\underset{s}{S}} \|\Psi_n(s) - \Psi_{n-1}(s)\|$$

From these estimates it is evident that the sequence $\{\Psi_n\}$ is Cauchy.

We have thus proven the following:

Theorem 5. Define $\Psi_n(c, t)$ by the iterative scheme, Eqs. (16). Then for t sufficiently small, $\Psi_n(t)$ converges in B to $\Psi(t)$, and Ψ is a solution of Eq. (7).

From Theorem 1(a) and Theorem 5 we have the following:

Corollary 6. $\Psi(t) \in \mathscr{T}_+$.

4. GLOBAL SOLUTIONS

Proposition 4 and Theorem 5 demonstrate the existence of local solutions to the integral equations (6) and (7). Such solutions are often referred to as "mild" local solutions to the original equation. In this section we first extend the mild solutions to global solutions, i.e., solutions for all time, and then, by demonstrating differentiability, prove that the mild solutions are in fact strong solutions. As a by-product, we obtain unicity.

Theorem 6. Let Ψ_1, Ψ_2 be solutions of Eqs. (6), (7), respectively, satisfying $\Psi_1(0) = \Psi_2(0) = \varphi_0$. Then $\Psi_1 = \Psi_2$.

Proof. By using the evolution group property $T_\Psi(t, s)T_\Psi(s, r) = T_\Psi(t, r)$, $r \leqslant s \leqslant t$, we may compute from Eq. (7)

$$\begin{aligned} \Psi_2(t + s) &= T_{\Psi_2}(t + s, t)\left[T_{\Psi_2}(t, 0)\varphi_0 + \int_0^t T_{\Psi_2}(t, t')G(\Psi_2(t'))\, dt'\right] \\ &\quad + \int_t^{t+s} T_{\Psi_2}(t + s, t')G(\Psi_2(t'))\, dt' \\ &= T_{\Psi_2}(t + s, t)[\Psi_2(t) + sG(\Psi_2(t)) + O(s)] \end{aligned} \tag{17}$$

Let us define

$$\eta(s) = [T_{\Psi_2}(t + s, t) - U(s)]\xi_0$$

Computing

$$\begin{aligned}\partial\eta/\partial s &= -[A + D(\Psi_2(t + s))]T_{\Psi_2}(t + s, t)\xi_0 + AU(s)\xi_0 \\ &= -[A + D(\Psi_2(t + s))]\eta(s) - D(\Psi_2(t + s))U(s)\xi_0 \\ \eta(0) &= 0\end{aligned}$$

we have

$$\eta(s) = -\int_0^s T_{\Psi_2}(t + s, t + t')D(\Psi_2(t + t'))U(t')\,dt'\,\xi_0$$

Therefore,

$$\begin{aligned}T_{\Psi_2}(t + s, t)\xi_0 &= U(s)\xi_0 - T_{\Psi_2}(t + s, t)D(\Psi_2(t))\xi_0 + O(s) \\ &= U(s)\xi_0 - sU(s)D(\Psi_2(t))\xi_0 + O(s)\end{aligned}$$

Combining this with Eq. (17), we may write

$$\Psi_2(t + s) = U(s)[\Psi_2(t) + sG(\Psi_2(t)) - sD(\Psi_2(t))\Psi_2(t)] + O(s)$$

On the other hand,

$$\begin{aligned}\Psi_1(t + s) &= U(s)U(t)\varphi_0 + \int_0^t U(s)U(t - t')J(\Psi_1(t'))\,dt' \\ &\quad + \int_t^{t+s} U(t + s - t')J(\Psi_1(t'))\,dt' \\ &= U(s)[\Psi_1(t) + sJ(\Psi_1(t))] + O(s)\end{aligned}$$

Thus

$$\begin{aligned}\Psi_2(t + s) - \Psi_1(t + s) = U(s)\{&\Psi_2(t) - \Psi_1(t) \\ &+ s[J(\Psi_2(t)) - J(\Psi_1(t))]\} + O(s)\end{aligned}$$

Letting $p(t) = \|\Psi_2(t) - \Psi_1(t)\|$, we have immediately

$$\begin{aligned}p(t + s) - p(t) &\leqslant s\|J(\Psi_2(t)) - J(\Psi_1(t))\| \\ &\leqslant s\|J\|(\|\Psi_2(t)\| + \|\Psi_1(t)\|)\|\Psi_2(t) - \Psi_1(t)\|\end{aligned}$$

or

$$\mathscr{D}^+ p(t) \leqslant 2\|J\|mp(t)$$

where $m = \sup\{\|\Psi_i(t)\| \mid i = 1, 2, 0 \leqslant t \leqslant T_0\}$. By Gronwall's Lemma [15]

$$p(t) \leqslant p(0)e^{2\|J\|mt} = 0$$

completing the proof.

Corollary 7. The solution to Eqs. (6) and (7) is unique.

We now prove the solution of Eq. (6) is differentiable in t and thus is a solution of Eq. (1). Our proof utilizes Kato's Theorem,[16] which we paraphrase below:

Theorem (Kato). Let T be the generator of a holomorphic semigroup $U(t)$ on a Banach space X, and $f: \mathbb{R}_+ \to X$ a Hölder continuous function. Then the equation

$$d\varphi(t)/dt = -T\varphi(t) + f(t), \quad t > 0$$

subject to $\varphi(0) = f_0 \in X$, has the solution

$$\varphi(t) = U(t)f_0 + \int_0^t U(t-s)f(s)\,ds$$

and $\varphi(t)$ is continuously differentiable for $t > 0$.

Referring to Eq. (7), it is sufficient to show that $G(\Psi(t), \Psi(t)) - D(\Psi(t))\Psi(t)$ is Hölder continuous, since we already have shown [Theorem 1(b)] that $U(t) \equiv e^{-At}$ is a holomorphic semigroup. It is not too difficult to see that the Hölder continuity of $G(\Psi, \Psi) - D(\Psi)\Psi$ in t follows from that of Ψ, so we prove that Ψ is Hölder continuous.

Lemma 8. For $\varphi_0 \in \mathscr{T}_+ \cap \mathscr{D}(A)$, $\{\Psi_n(t)\}$, as specified by the iterative scheme, Eqs. (15), are differentiable on some interval $[0, T_0]$, and the derivatives $\{\Psi_n'(t)\}$ are uniformly bounded (in t and n).

Proof. To compute the derivative we estimate

$$\begin{aligned}\Psi_n(t+h) - \Psi_n(t) &= (U(h) - I)\Psi_n + hU(h)J(\Psi_{n-1}(t)) + O(h)\\ &= (U(h) - I)\Big\{U(t)\varphi_0\\ &\quad + \int_0^t U(t-t')[J(\Psi_{n-1}(t')) - J(\Psi_{n-1}(t))]\,dt'\\ &\quad + \int_0^t U(t-t')J(\Psi_{n-1}(t))\,dt'\Big\}\\ &\quad + hU(h)J(\Psi_{n-1}(t)) + O(h)\end{aligned}$$

Therefore the right derivative $\mathscr{D}^+\Psi_n$ is

$$\mathscr{D}^+\Psi_n(t) = AU(t)\varphi_0 + A\int_0^t U(t-t')[J(\Psi_{n-1}(t))]\,dt' + U(t)J(\Psi_{n-1}(t))$$

where we have used the identity

$$A\int_0^t U(t-t')\,dt' = U(t) - I$$

Then a bound on $\mathscr{D}^+\Psi_n(t)$ is obtained by the estimate (Ref. 14, p. 489) $\|AU(t)\| \leqslant K/t$ for analytic semigroups, where K is some constant. Thus for $n \geqslant 2$,

$$\begin{aligned}\|\mathscr{D}^+\Psi_n(t)\| &\leqslant \|A\varphi_0\| + t\|AU(t-t')\|\,\|J(\Psi_{n-1}(t')) - J(\Psi_{n-1}(t))\| \\ &\quad + \|J(\Psi_{n-1}(t))\| \\ &\leqslant \|A\varphi_0\| + tK\|J\|2m\frac{\|\Psi_{n-1}(t') - \Psi_{n-1}(t)\|}{t'-t} + \|J\|m^2 \qquad (18)\end{aligned}$$

and a uniform bound is obtained inductively by estimating Lipschitz constants K_n for the $\{\Psi_n\}$. Indeed, for $n = 1$,

$$\begin{aligned}\mathscr{D}^+\Psi_1(t) &= AU(t)\varphi_0 + A\int_0^t U(t-t')J(\varphi_0)\,dt' + J(\varphi_0) \\ &= U(t)A\varphi_0 + U(t)J(\varphi_0)\end{aligned}$$

Thus Ψ_1 is Lipschitz with constant $K_1 = \|A\varphi_0\| + \|J\|\,\|\varphi_0\|^2$. Using the differentiability of Ψ_n and the estimate (18) for $\|\mathscr{D}^+\Psi_n\|$, we have

$$\|\Psi_n(t) - \Psi_n(s)\| \leqslant (\|A\varphi_0\| + \|J\|m^2 + 2tKm\|J\|K_{n-1})|t-s|$$

so that

$$K_n = \alpha + t\beta K_{n-1}; \qquad \alpha = \|A\varphi_0\| + \|J\|m^2, \qquad \beta = 2Km\|J\|$$

which is uniformly bounded for $t < 1/(2Km\|J\|)$. This proves the theorem.

Corollary 9. The $\{\Psi_n(t)\}$ as specified by the iterative scheme, Eqs. (15), are Lipschitz with a uniform Lipschitz constant $K_\infty = \alpha(1 - T_0\beta)^{-1}$ for $t < T_0$.

We now state the following result:

Theorem 10. For $\varphi_0 \in \mathscr{T}_+ \cap \mathscr{D}(A)$, the solution of Eq. (6) is differentiable, and therefore a solution of Eq. (1), for $t < T_0$.

Proof. All of the conditions of Kato's Theorem, as stated above, are satisfied.

Our final step is to extend our solution $\Psi(t)$ to a global solution.

Proposition 11. For $\phi_0 \in \mathscr{T}_+ \cap \mathscr{D}(A)$, $\|\Psi(t)\| = \|\varphi_0\|$ for t sufficiently small that $\Psi(t)$ obeys both Eq. (6) and Eq. (7).

Proof. Integrate Eq. (6) over c and sum over the spatial index i. Recalling $\Psi \in \mathscr{T}_+$ and $e^{-tA} = U(t)$ is an isometry on $\mathbb{R}_+$, we obtain

$$\|\Psi\| = \|\varphi_0\| + \int dc\int_0^t ds \sum_{i=1}^{n^3} \{U(s)[G(\Psi(s) - D(\Psi(s))\Psi(s)]_i\, ds$$

Using Corollary 2, this becomes

$$\|\Psi\| = \|\varphi_0\| + \int_0^t ds \sum_{i=1}^{n^3} \int dc\, [G(\Psi(s)) - D(\Psi(s))\Psi(s)]$$

But $\int dc\, [G(\Psi(s)) - D(\Psi(s))\Psi(s)] = 0$, since 1 is a collision invariant. This proves Proposition 11.

But now, the global property of the solution $\Psi(t)$ follows immediately, for the solution can be obtained in sufficiently small time steps, and this procedure, by virtue of Proposition 11, can be continued ad infinitum. We collect all of the above results as the following theorem:

Theorem 12. Suppose $\varphi_0 \in \mathscr{T}_+ \cap \mathscr{D}(A)$. Then there exists a unique positive solution $\Psi(t)$ of the integral equation (6)—or, equivalently, Eq. (7)—for all $t \geqslant 0$, and (i) $\Psi(t)$ is a continuously differentiable solution of the Boltzmann equation (1) for all $t > 0$; (ii) $\Psi(t) \in \mathscr{T}_+$, $t \geqslant 0$; (iii) Ψ depends continuously upon the initial datum φ_0.

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